Taylor's Theorem

Taylor polynomials and Lagrange error bounds

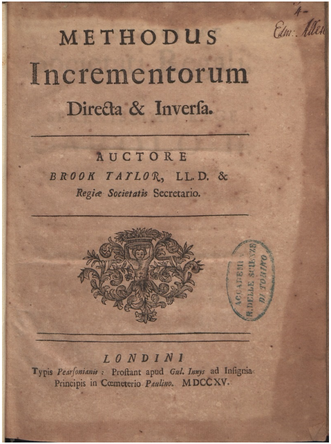
Edward Pearce

The University of Sheffield

Wednesday 16th December

Historical note

Brook Taylor (1685-1731)



METHODUS
Incrementorum
Directa & Inverſa.

AUCTORE
BROOK TAYLOR, LL. D. &
Regiæ Societatis Secretario.

LONDINI
Typis Pearſonianis: Proſtant apud Gul. Innys ad Inſignia
Principis in Cœmeterio Paulino. MDCCXV.

Direct and Reverse Methods of Incrementation (1715)

Motivation

Question

How do we know $e \approx 2.71828...$?

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Polynomials are easy to compute, take derivatives, integrate...

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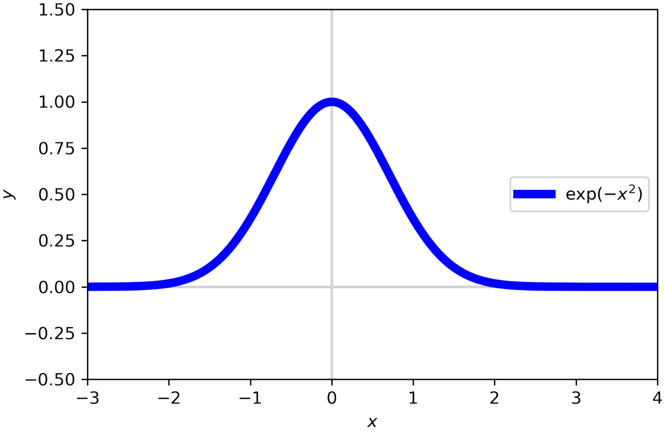
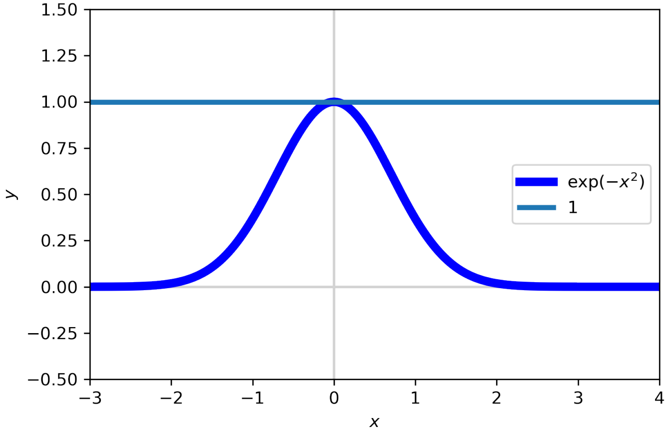
Polynomials are easy to compute, take derivatives, integrate...
We can approximate a k-times differentiable function around a given point by a polynomial of degree k.

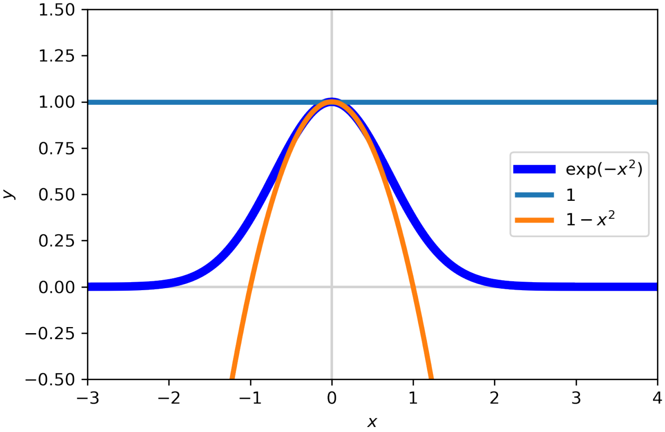
Example: Density of Normal Distribution



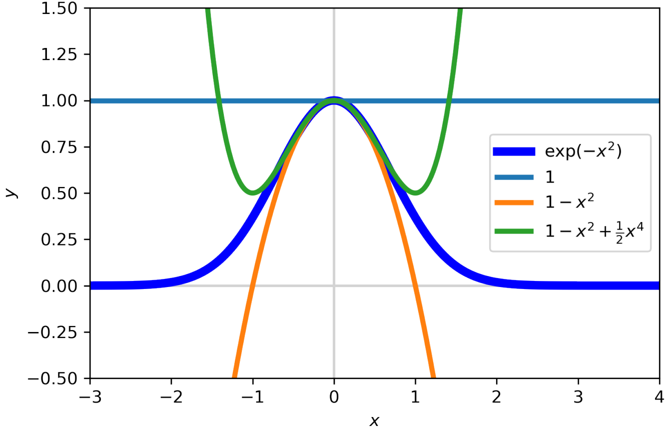
Linear approximation to bell curve



Quadratic approximation to bell curve



Polynomial approximations to bell curve



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Definition

Let $k \geq 1$ be an integer and let $f : \mathbb{R} \to \mathbb{R}$ be a k times differentiable at the point $a \in \mathbb{R}$. Define the k-th Taylor polynomial of the function f at the point a to be

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

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Remark

The k-th Taylor polynomial $P_k(x)$ of the function f at the point a is defined such that $P_k^{(j)}(a) = f^{(j)}(a)$ for all integers $0 \leq j \leq k$.

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Example (Linear approximation)

Near $x = a$, $f(x) \approx f(a) + f'(a)(x - a)$

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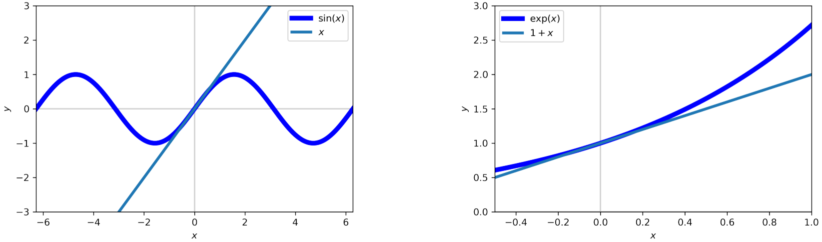

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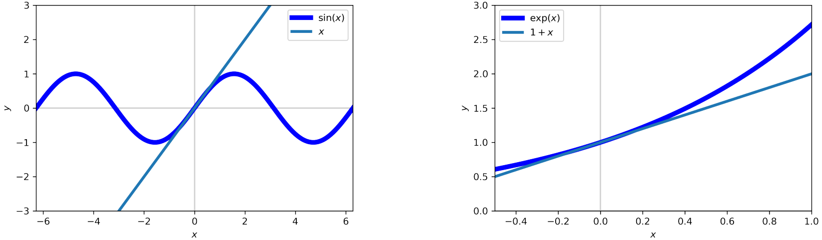

The graph of $y = P_1(x)$, approximating the function f near a, is the tangent line to the graph $y = f(x)$ at $x = a$.

Special cases 2

Example (Quadratic approximation)

Sufficiently close to $x = a$, a more accurate approximation is
$f(x) \approx P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$

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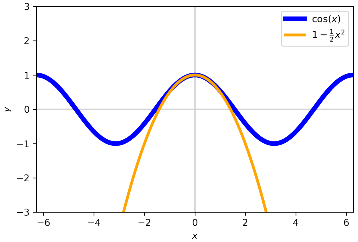
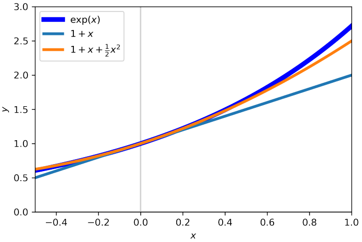
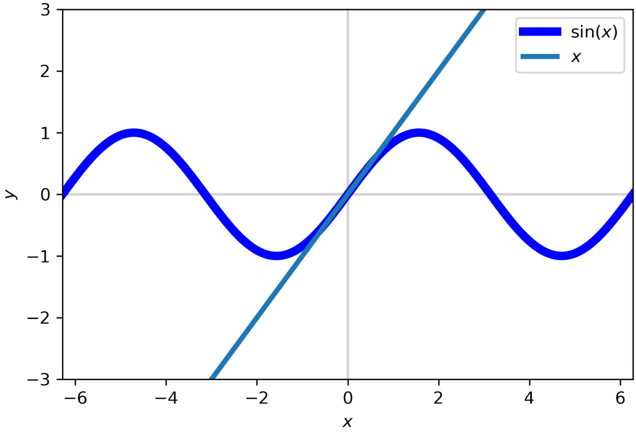
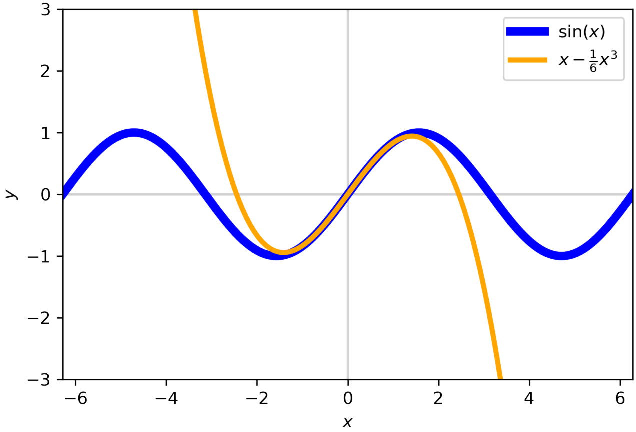




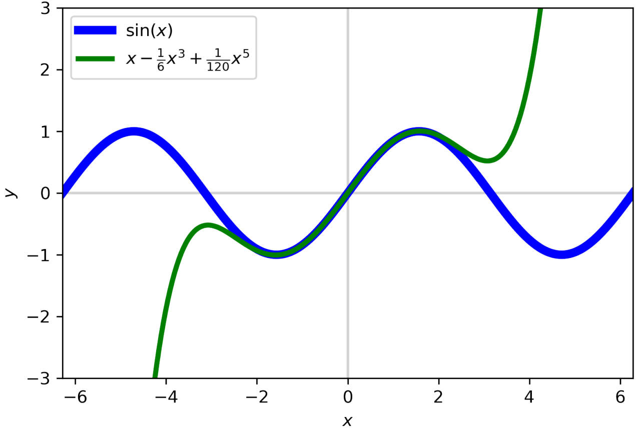
More terms/higher order approximations, ...



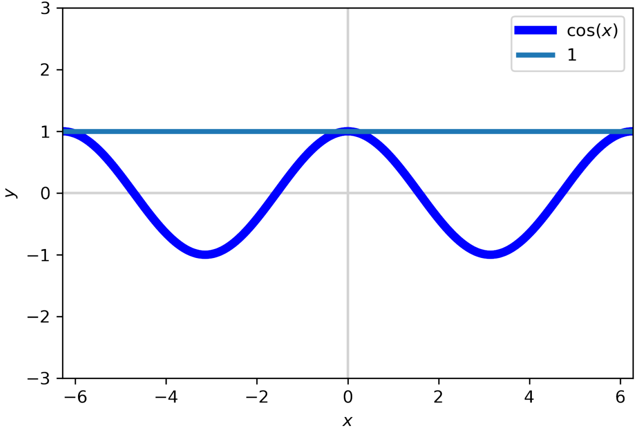
More terms, more accuracy, ...



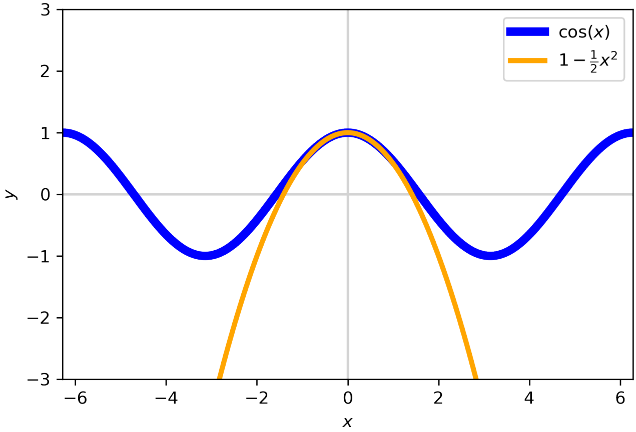
How accurate, how fast?



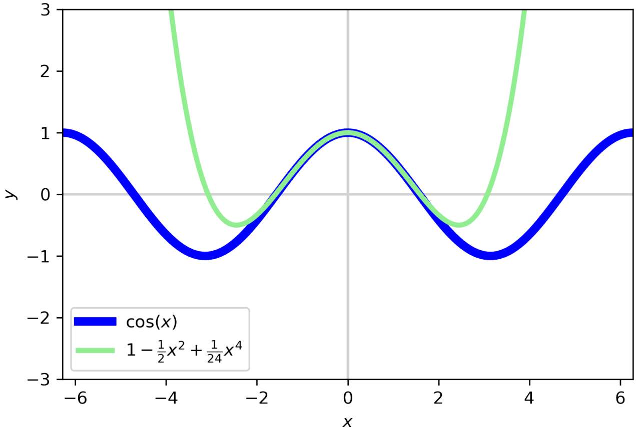
How many terms to bound error (decimal places)?



Max error for given interval and polynomial degree?



Size of interval with given error tolerance?



Taylor's theorem

Theorem (Taylor's theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ *be a* k *times differentiable at the point* $a \in \mathbb{R}$, *and let* $P_k(x)$ *be the* k*-th Taylor polynomial of* f *at* a.
Then the error/remainder $R_k(x) = f(x) - P_k(x)$ *between the function* f *and its* k*-th Taylor polynomial can be expressed in the form:*

$$R_k(x) = h_k(x)(x - a)^k$$

for a function $h_k : \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{x \to a} h_k(x) = 0$.

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Let $f : \mathbb{R} \to \mathbb{R}$ be a k times differentiable at the point $a \in \mathbb{R}$, and let $P_k(x)$ be the k-th Taylor polynomial of f at a. Then the error/remainder $R_k(x) = f(x) - P_k(x)$ between the function f and its k-th Taylor polynomial can be expressed in the form:

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Interpretation

As $x \to a$, the error term $R_k(x) = f(x) - P_k(x)$ tends to 0 faster than $(x-a)^k$, the highest order term of $P_k(x)$, so $P_k(x)$ is the "asymptotic best fit" degree k polynomial to f at a.

Lagrange form for Taylor approximation error

Theorem (Mean-value form of the remainder)

Let $f : \mathbb{R} \to \mathbb{R}$ be a $k+1$ times differentiable on (a, x) with $f^{(k)}$ continuous on $[a, x]$. Then the error/remainder $R_k(x) = f(x) - P_k(x)$ can be expressed as:

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

for some real number $c \in (a, x)$. Similarly, when $x < a$.

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For the proof, we first recall the mean value theorem.

Theorem (Mean Value Theorem (MVT))

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b), where $a < b$. Then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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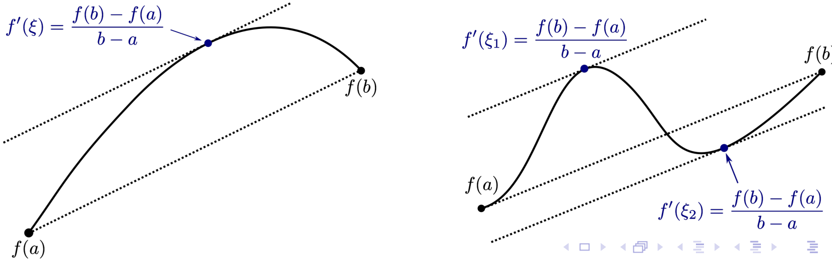

Proof of Taylor's theorem (explicit remainder)

For fixed a and x, consider the function $F(t) : [a, x] \to \mathbb{R}$ constructed such that $F(x) = f(x)$ and $F(a) = P_k(x)$ given by

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which also satisfies the criteria for MVT by the assumptions on f. Note that $F(x) - F(a) = f(x) - P_k(x) = R_k(x)$. We compute the derivate of F with respect to t using the product rule and chain rule, and note a telescoping cancellation of terms such that

$$F'(t) = \frac{f^{(k+1)}(t)}{k!}(x-t)^k$$

Proof of Taylor's theorem (part 2)

Now consider the function $H(t) : [a, x] \to \mathbb{R}$ given by

$$H(t) = F(t) + \frac{F(x) - F(a)}{(x-a)^{k+1}}(x-t)^{k+1}$$

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which also satisfies the criteria for MVT, with $H(x) = H(a)$. Applying the MVT (Rolle's theorem) to $H(t)$, there exists $c \in (a, x)$ such that

$$H'(c) = \frac{f^{(k+1)}(c)}{k!}(x-c)^k - (k+1)\frac{R_k(x)}{(x-a)^{k+1}}(x-c)^k = 0$$

Rearranging this gives the desired formula for $R_k(x)$. □

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Throughout the previous proof, we were treating a and x as fixed constants and instead using t as the independent variable when applying the Mean Value Theorem.

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Throughout the previous proof, we were treating a and x as fixed constants and instead using t as the independent variable when applying the Mean Value Theorem.
If we decided to apply the MVT directly to F rather than to H we would have obtained an alternative formula for the Taylor approximation error (called the Cauchy form):

$$R_k(x) = \frac{f^{(k+1)}(c)}{k!}(x-c)^k(x-a)$$

for some real number $c \in (a, x)$ (a different constant than in the Lagrange form of the remainder).

Applications

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How many terms in a Taylor polynomial approximation do we need to bound the error to a certain number of decimal places?

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Theorem (Estimating Taylor approximation error)

Suppose f is $(k+1)$ times continuously differentiable on $[a-r, a+r]$ and $|f^{(k+1)}(x)| \leq M$ for all $x \in (a-r, a+r)$ (some $r > 0$). Then we can bound the error

$$|R_k(x)| = \frac{|f^{(k+1)}(c)|}{(k+1)!}|x-a|^{k+1} \leq M\frac{|x-a|^{k+1}}{(k+1)!} \leq M\frac{r^{k+1}}{(k+1)!}$$

for all $x \in (a-r, a+r)$.

Examples

For $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$, $|\cos(x) - (1 - \frac{1}{2}x^2)| < \frac{\pi^4}{24 \times 4^4} \approx 0.016..$

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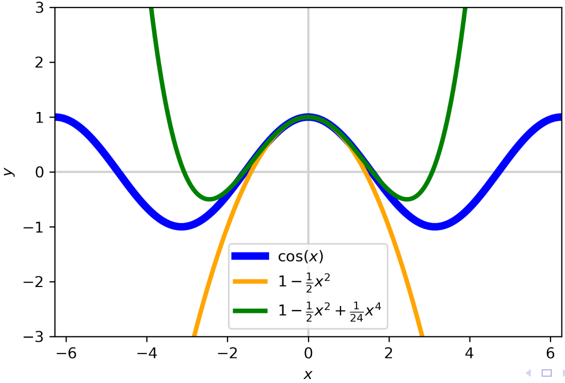

We may calculate $e \approx 2.71828$, correct up to five decimal places, using the fact that for $-1 \leq x \leq 1$

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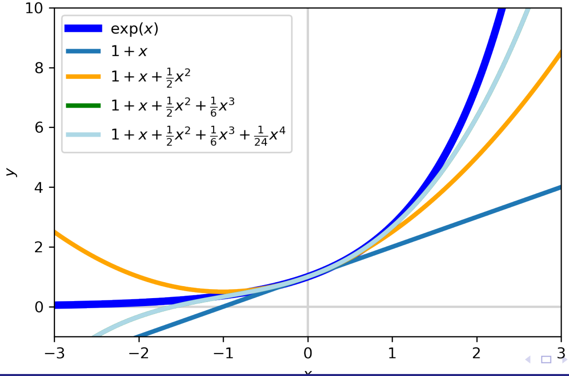

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- We can approximate differentiable functions by polynomials
- We can calculate an upper bound on the error between our approximations to a function and its true value
- We understand that linear and quadratic approximations have practical uses in physics and engineering, but may diverge outside a neighbourhood of the approximation point

Future Directions

- Examples of functions where Taylor's theorem does not apply (e.g. antiderivative of $\sin(\frac{1}{x})$)

Image Credits: Wikipedia (Brook Taylor, Mean Value Theorem); Matplotlib

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- Applications in numerical analysis (finite difference methods)

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End

Thanks for listening!