Hilbert schemes of Cyclic Quotient Surfaces

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Introduction

If Y is a cyclic quotient singularity, then the topology of the Hilbert scheme $\mathrm{Hilb}^n(Y)$ may be understood in terms the intersection pairing on the exceptional curves in the maximal resolution of Y. In an algebraic analogue to Morse theory, we use a Białynicki-Birula decomposition on $\mathrm{Hilb}^n(Y)$ to determine its topological properties. The critical points and their indices are computed using the combinatorics of partitions. We may analyse $\mathrm{Hilb}^n(Y)$ from only the 'core' partitions, and their structure is governed by the exceptional curves of the maximal resolution of Y.

Hilbert schemes of points on smooth surfaces

Let $X^{[n]} = \mathrm{Hilb}^n(X)$ be the Hilbert scheme of n points on a smooth quasiprojective surface X over $\mathbb{C}$. Then $X^{[n]}$ is a smooth $2n$-dimensional complex manifold parametrising n points on X. When $X = \mathbb{C}^2$ is the complex plane, points in $(\mathbb{C}^2)^{[n]}$ are simply codimension n ideals in $\mathbb{C}[x,y]$.
Ellingsrud-Strømme exploited the action of an algebraic torus on $\mathbb{C}^2$ to derive the generating function for the Poincaré polynomials $P_t(X^{[n]})$ of $X^{[n]}$ in the case $X = \mathbb{C}^2$, reducing the problem to computations in the combinatorics of partitions. $(\mathbb{C}^\times)^2$-invariant points in $(\mathbb{C}^2)^{[n]}$ are precisely the codimension n monomial ideals in $\mathbb{C}[x,y]$, and these are in bijection with partitions of n. Göttsche generalised this result to any smooth surface X showing that the homology of $X^{[n]}$ is determined by the homology of X. More precisely, if $b_i(X)$ is the ith Betti number of X, then

$$\sum_{n=0}^{\infty} q^n P_t(X^{[n]}) = \prod_{m=1}^{\infty}\prod_{i=0}^{4}\left(1+(-1)^{i+1}t^{2m-2+i}q^m\right)^{(-1)^{i+1}b_i(X)}$$

Can we obtain similar insight into $\mathrm{Hilb}^n(X)$ when we allow X to have mild singularities?

Hilbert schemes of quotient singularities

If G is a finite Abelian group acting on $\mathbb{C}^2$with trivial stabilisers away from the origin, then $\mathbb{C}^2/G$ is a surface whose smooth points are free G-orbits of points in $\mathbb{C}^2$ and which has an isolated singularity at the origin. Points in $\mathrm{Hilb}^n(\mathbb{C}^2/G)$ are G-invariant codimension n ideals in $\mathbb{C}[x,y]$, thus give rise to n-dimensional representations of G. $\mathbb{C}^2/G$ also admits a torus action, so Ellingsrud-Strømme's method may be used to understand the homology of $\mathrm{Hilb}^n(\mathbb{C}^2/G)$. $\mathrm{Hilb}^n(\mathbb{C}^2/G)$ splits into connected components $\mathrm{Hilb}_v(\mathbb{C}^2/G)$ consisting of ideals giving rise to the same n-dimensional G-representation v. The G-Hilbert scheme of $\mathbb{C}^2$ is the component of $\mathrm{Hilb}^{|G|}(\mathbb{C}^2/G)$ corresponding to the regular representation of G, which is the minimal resolution for $\mathbb{C}^2/G$ via the Hilbert-Chow map.

Białynicki-Birula decomposition

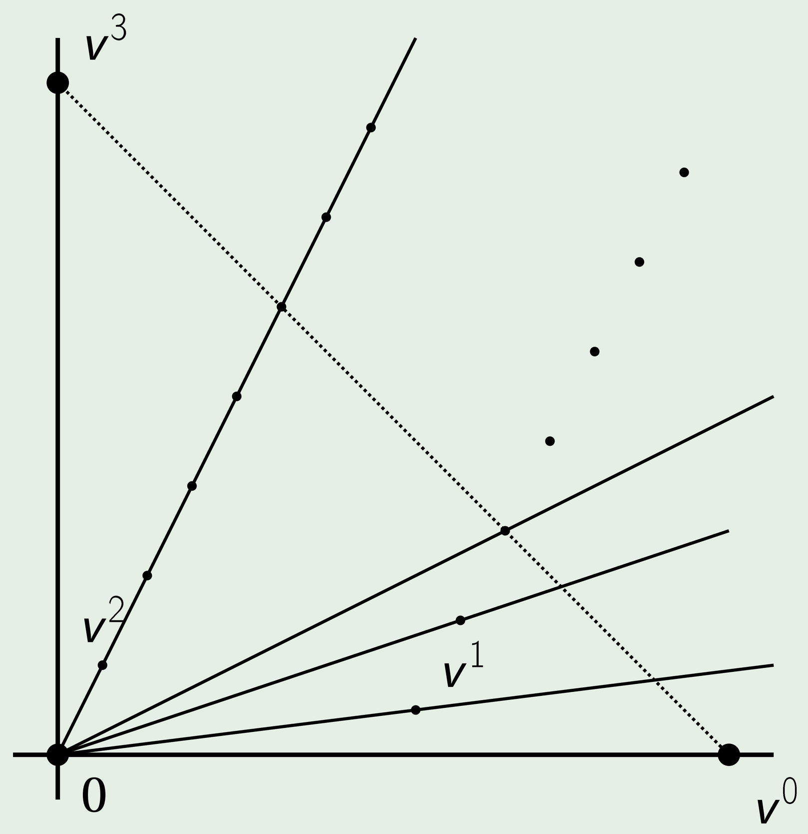
Comparable to Morse theory, the Białynicki-Birula decomposition expresses a space with a torus action as a union of affine spaces indexed by the fixed points of the action. If we know the dimension of the affine space associated to each fixed point, then we may understand the topology of the overall space. Each connected component $\mathrm{Hilb}_v(\mathbb{C}^2/G)$ admits an induced toric action, so we may understand the topology of $\mathrm{Hilb}^n(\mathbb{C}^2/G)$ through this decomposition. The index of a fixed point is determined by the weights of the torus action on its tangent space, which may be computed as certain statistics on the corresponding partition.

Maximal Resolutions and Toric Geometry

For a resolution $\pi : Y' \to Y$ of a surface quotient singularity Y we may write $K_{Y'|Y} := K_{Y'} - \pi^* K_Y = \sum_j (\alpha_j - 1)E_j$, where E_j are the exceptional divisors, $\alpha_j \in \mathbb{Q}$. In the study of deformation theory, Kollár and Shepherd-Barron defined the maximal resolution in terms the discrepencies: π is called maximal if it is maximal with respect to the property $0 < \alpha_j < 1$.
The maximal resolution is uniquely determined, and can be constructed from the minimal resolution by successive blowing up of points $E_i \cap E_j$ with $\alpha_i + \alpha_j \geq 0$.
When we have a cyclic quotient singularity (i.e. a two-dimensional toric variety) Y_σ, the minimal resolution of Y_σ is the toric variety corresponding to the subdivision of σ by rays through points on the Newton boundary of σ, whereas the maximal resolution of Y_σ is the toric variety obtained by subdividing the polyhedral cone $\sigma = \left\langle v^0; v^{k+1}\right\rangle \subset \mathbb{R}^2$ by drawing rays through 0 and all interior lattice points of the triangle $\Delta := \mathrm{conv}(0, v^0, v^{k+1})$, respectively.

Diagram: Resolutions of $Y(15,2)$

The maximal resolution is given by the polyhedral subdivision Σ of $\sigma = \langle v^0; v^3\rangle$ given below, whilst the minimal resolution uses only the rays through v^1 and v^2, respectively.



Hirzebruch-Jung Continued fractions

Let r, b be coprime integers with $r > b > 0$. Then the Hirzebruch-Jung continued fraction of r/b is the expression

$$\frac{r}{b} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}} = [a_1, a_2, \dots, a_k]$$

This is a unique expression for any rational number greater than 1, and has all $a_i \geq 2$.

Continued fractions and the minimal resolution

For $Y_\sigma = Y(r,b)$ we may label points on the Newton boundary of $\sigma = \langle (1,0); (-b,r)\rangle = \left\langle v^0; v^{k+1}\right\rangle$ by $v^0, \dots, v^{k+1}$. The minimal resolution of Y_σ is the toric variety Y_Σ corresponding to the polyhedral subdivision Σ of σ by rays through the origin and each point v^j, respectively. We have relations $v^{j-1} + v^{j+1} = a_j v^j$ for $j = 1, \dots, k$, where $\frac{r}{b} = [a_1, a_2, \dots, a_k]$. Moreover, the numbers $-a_j$ are equal to the self intersection numbers of the exceptional divisors. The maximal resolution can be constructed from the minimal resolution by successive blowing up of certain points $E_i \cap E_j$, and the self intersection numbers of the resulting exceptional divisors may also be obtained by an analogous procedure on the continued fraction.

Core partitions

For a cyclic quotient singularity Y, we perform a Białynicki-Birula decomposition of $\mathrm{Hilb}^n(Y)$ using the torus action induced from the action on $\mathbb{C}^2$. The torus-fixed points correspond to partitions of n, and the torus weights on the tangent space to a fixed point may be computed as certain statistics on the corresponding partition. If we impose certain restrictions on these partition statistics we obtain a class called the core partitions, which generate all the others. For these core partitions, the tangent space to the associated fixed point in $\mathrm{Hilb}^n(Y)$ is restricted so that the associated cell is stuck over the singular point in Y. The core partitions form a lattice governed by the exceptional curves of the maximal resolution of Y, and the area of a core partition is a quadratic form on the coordinates whose purely quadratic part may be obtained from the intersection pairing on the exceptional curves.

Theorem

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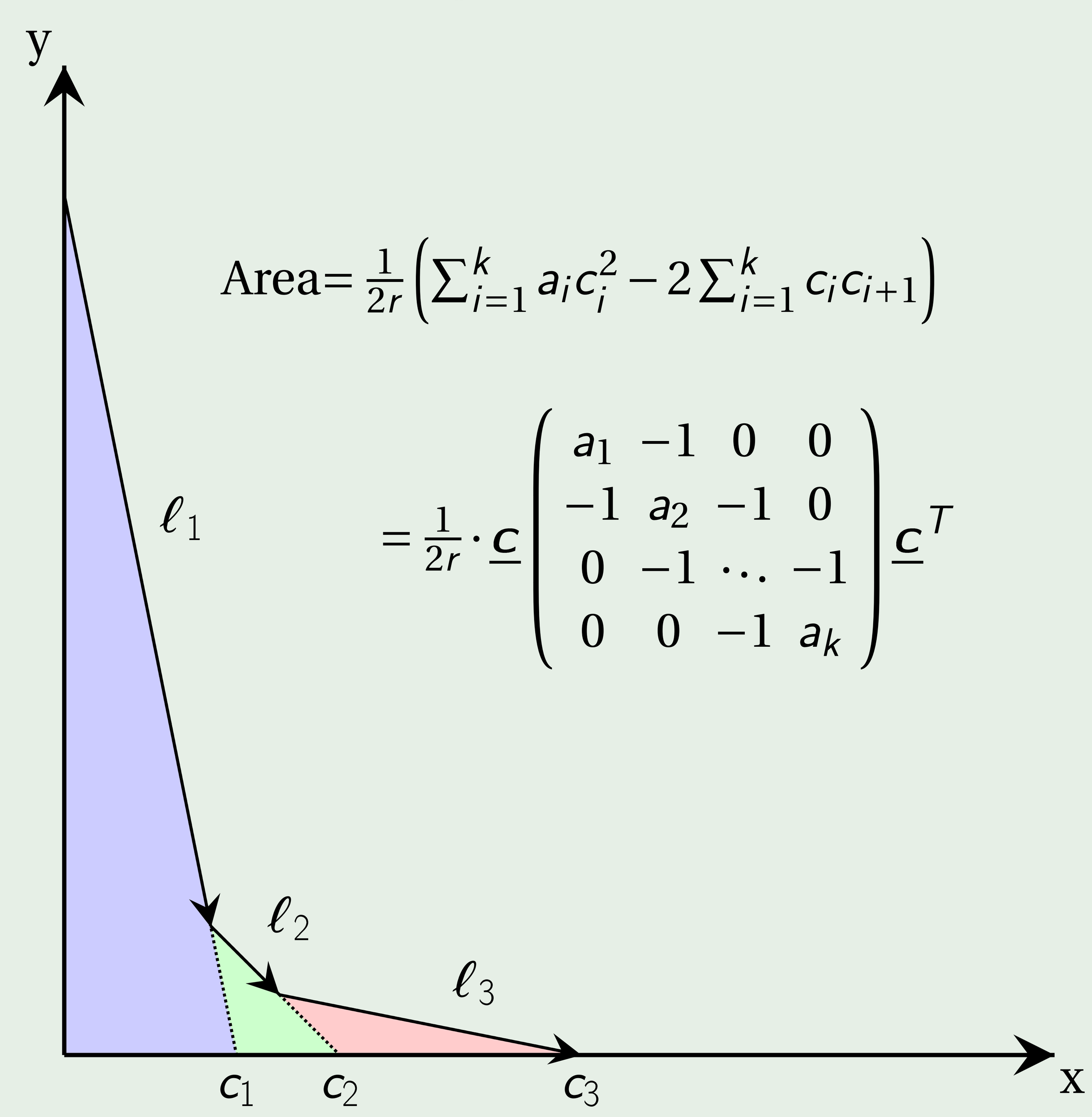
Diagram: Linear approximation of a core partition

The core partitions may be split into parts made up of repeating segments, where the average slope of the basic segments are given by the primitive lattice points generating the rays in the maximal resolution. The diagram below illustrates the macroscopic structure of core partitions by approximating the parts of the core partition by lines with their average slope.



The area under the configuration of lines in terms of the coordinates of the intercepts of the lines is a quadratic form which may also be obtained from the intersection pairing on the exceptional curves on the maximal resolution. The discreteness of partitions introduces additional linear terms into their exact area formula.

Conclusion

If Y is a cyclic quotient singularity, then the topology of the Hilbert scheme $\mathrm{Hilb}^n(Y)$ may be understood in terms the intersection pairing on the exceptional curves in the maximal resolution of Y. Next steps could include finding Göttsche-type product formulas for the Betti numbers.